MIXING OF CARTESIAN SQUARES OF POSITIVE OPERATORS

BY

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ABSTRACT

Let T be a power bounded positive operator in $L_1(X, \Sigma, m)$ of a probability space, given by a transition measure $P(x, A)$. The Cartesian square S is the operator on $L_1(X \times X, \Sigma \times \Sigma, m \times m)$ induced by the transition measure $Q((x, y), A \times B) = P(x, A)P(y, B)$. T is *completely mixing* if $\int u\, e\, dm = 0$ implies $T^n u \to 0$ weakly (where $0 \leqq e \in L_\infty$ with $T^* e = e$).

Theorem. If T has no fixed points, then T is completely mixing if and only if S is completely mixing.

1. Definitions and notation

Let (X, Σ, m) be a probability space and let T be a positive operator on $L_1(X,\Sigma,m)$, (hence T is bounded). We consider here T *power bounded*, i.e. $\sup \| T^n \| = K < \infty$. For such an operator, Sucheston [7] has proved that X decomposes into two disjoint sets, the *remaining part* Y and the *disappearing part* Z, such that $\| T^n u \|_1 \to 0$ for every $u \in L_1(Z)$, while there exists a function $e > 0$ a.e. on Y with $T^* e = e$ (hence $\liminf \| T^n u \|_1 > 0$ for $0 \leqq u \in L_1(Y)$, $u \not\equiv 0$).

A function $0 \not\equiv u \in L_1(X,\Sigma,m)$ is a *fixed point* for T if $Tu = u$. By the decomposition $u = u^+ - u^-$ we have $u^+ - u^- = Tu = Tu^+ - Tu^-$ so $Tu^+ \geqq u^+$ and $\lim T^n u^+ \in L_1$ is a fixed point, and the same applies to $\lim T^n u^-$, so it is enough to consider the existence of non-negative fixed points.

In this paper we relate the convergence properties of the powers of T to those of the powers of its Cartesian square (defined below).

We start by generalizing a result of [5]:

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THEOREM 1.1. *If T has no non-zero fixed point, then the weak convergence of $T^n u$ implies $\| T^n u \|_1 \to 0$.*

PROOF. Define $v_n = | T^n u |$. As $T^n u$ converges weakly, it is uniformly absolutely continuous with respect to m, hence $\{v_n\}$ is uniformly absolutely continuous. $\pm T^n u \leqq v_n$ so $0 \leqq v_{n+1} \leqq T v_n$. Define a functional on L_∞ by a Banach limit:

$$\nu(f) = \mathrm{LIM}\left\{\int v_n f\, dm\right\}.$$

The uniform absolute continuity of v_n implies that ν is a finite measure, i.e. $\nu(f) = \int v f\, dm$ for some $0 \leqq v \in L_1$. For $0 \leqq f \in L_\infty$ we obtain

$$\langle Tv, f\rangle = \langle v, T^* f\rangle = \mathrm{LIM}\{\langle v_n, T^* f\rangle\} = \mathrm{LIM}\{\langle Tv_n, f\rangle\}$$
$$\geqq \mathrm{LIM}\{\langle v_{n+1}, f\rangle\} = \langle v, f\rangle.$$

Hence $Tv \geqq v$, and $T^n v$ is increasing. If $w = \lim T^n v$, then by the monotone convergence theorem $\int w = \lim \int T^n v \leqq K \| v \|_1$ so $0 \leqq w \in L_1$ and $Tw = w$, so by the nonexistence of fixed points $w \equiv 0$ and $v = 0$. Thus $\mathrm{LIM}\{ \int v_n dm\} = 0$. This implies $\liminf \| T^n u \|_1 = 0$. But

$$\| T^m u \|_1 < \varepsilon \Rightarrow \| T^{m+n} u \|_1 \leqq K\varepsilon \text{ so } \| T^n u \|_1 \to 0.$$

In this note we assume that T is induced by a transition measure $P(x, A)$, i.e. $f \in L_\infty \Rightarrow T^* f(x) = \int f(y) P(x, dy)$ ma.e.

For $x, y \in X$ and $A, B \in \Sigma$ we define $Q((x,y), A \times B) = P(x, A)P(y, B)$, which can be uniquely extended to a transition measure on $(X \times X, \Sigma \times \Sigma)$. We denote by S the positive operator induced on $L_1(X \times X, \Sigma \times \Sigma, m \times m)$: the *Cartesian square* of T. (Even without transition measures, $S = T \otimes T$, the tensor product operator in $L_1(m) \otimes L_1(m)$).

LEMMA 1.1. (a) *If $h(x,y) = f(x)g(y)$ $(f, g \in L_\infty(m))$ then $S^n h(x,y) = T^{*n} f(x) T^{*n} g(y)$. In any case $S^{*n} h(x,y) = \int \int h(s,t) P^n(x, dt) P^n(y, ds)$.*

(b) *If $w(x,y) = u(x)v(y)$ $(u, v \in L_1(m))$ then $S^n w(x,y) = T^n u(x) T^n v(y)$.*

(c) *S is power bounded.*

(a) follows from Fubini's theorem. To prove (b) we use Fubini's theorem and the extension theorem. (c) follows from (a).

LEMMA 1.2. *The remaining part of S is $Y \times Y$.*

PROOF. Let $\tilde{e}\,(x,y) = e(x)e(y)$ where $T^* e = e$ and $e > 0$ a.e. on Y. Then $S^* \tilde{e} = \tilde{e}$ and $\tilde{e} > 0$ a.e. on $Y \times Y$ so $Y \times Y$ is in the remaining part. Let

$0 \leqq u(x) \in L_1$ have support in Z and take $0 < v(x) \in L_1$. By Lemma 1.1 (b) we have for $w_1(x,y) = u(x)v(y)$ and $w_2(x,y) = v(x)u(y)$ that $\| S^n w_i \| = \| T^n u \| \, \| T^n v \| \to 0$ so w_i must be supported in the disappearing part of S. As they are supported on $Z \times X$ and $X \times Z$, we have that $X \times X - Y \times Y$ is in the disappearing part.

2. Complete mixing of the Cartesian square

DEFINITION 2.1. Let T be a power bounded positive operator and $e > 0$ a.e. on Y with $T^* e = e$. T is *completely mixing* if $\int u\, e\, dm = 0$ implies $T^n u \to 0$ weakly in L_1.

REMARK. By the Hahn-Banach theorem, if T is completely mixing then e is uniquely defined (up to a multiplicative constant).

THEOREM 2.1. *Let T be power bounded having no fixed point. Then T is completely mixing if and only if its Cartesian square S is completely mixing.*

PROOF. The case $m(Y) = 0$ being trivial, we assume $m(Y) > 0$. If $T^* e = e$ with $e > 0$ on Y we denote $e'(x,y) = e(x)e(y)$, and $S^* e' = e'$ by Lemma 1.1.

We first show that if S is completely mixing so is T. The unique S^*-invariant function is e'. Let $u \in L_1(m)$ satisfy $\int u \; e\, dm = 0$. Define $w(x,y) = u(x)u(y)$, so $\int \int w e' \, d(m \times m) = 0$ by Fubini's theorem. For $f \in L_\infty(m)$ $F(x,y) = f(x)f(y)$ is in $L_\infty(m \times m)$ so by Lemma 1.1 and the complete mixing of S we have

$$|\langle T^n u, f\rangle|^2 = \int T^n u(x) f(x) m(dx) \int T^n u(y) f(y) m(dy)$$

$$= \int\int S^n w(x,y) F(x,y) d(m \times m) = \langle S^n w, F\rangle \to 0$$

Hence $\langle T^n u, f\rangle \to 0$ for every $f \in L_\infty$ and T is completely mixing.

We assume now that T is completely mixing. Let $w(x,y) \in L_1(m \times m)$ satisfy $\int \int w\, e' \, d(m \times m) = 0$ and define $u(x) = \int w(x,y) e(y) m(dy)$. By Fubini's theorem $u \in L_1(m)$ with $\int u \; e \; dm = 0$.

For $f \in L_\infty(m \times m)$ define $g_n(x) = \int \int f(x,s) P^n(y, ds) m(dy)$. $g_n \in L_\infty(m)$ by Fubini's theorem (we take an everywhere bounded representative of f).

$$\int\int u(x)\, S^{*n} f(x,y) m(dy) m(dx) = \int u(x) \left[\int S^{*n} f(x,y) m(dy) \right] m(dx)$$

$$= \int u(x) \left[\int\int\int f(t,s) P^n(x,dt) P^n(y,ds) m(dy) \right] m(dx)$$

$$= \int u(x) \left[\int \left\{ \int \int f(t,s) P^n(y,ds) m(dy) \right\} P^n(x,dt) \right] m(dx) = \int u(x) T^{*n} g_n(x) m(dx).$$

As $\| g_n \|_\infty \leqq K \| f \|_\infty$ for every n, we have by Theorem 1.1 (the nonexistence of fixed points implies $\| T^n u \|_1 \to 0$):

$$\left| \int \int u(x) S^{*n} f(x,y) m(dy) m(dx) \right| = |\langle T^n u, g_n \rangle| \leqq \| T^n u \|_1 K \| f \|_\infty \to 0.$$

$$(*) \quad \left| \int \int w(x,y) S^{*n} f(x,y) m(dx) m(dy) \right| \leqq \left| \int \int u(x) S^{*n} f(x,y) m(dx) m(dy) \right|$$

$$+ \left| \int \int [w(x,y) - u(x)] S^{*n} f(x,y) m(dx) m(dy) \right|.$$

We have already shown that the first term tends to zero.

For fixed x define $h_{nx}(y) = \int f(t,y) P^n(x,dt)$ which is measurable in y by Fubini's theorem, and we have

$$T^{*n} h_{nx}(y) = \int h_{nx}(s) P^n(y,ds) = \int \int f(t,s) P^n(x,dt) P^n(y,ds)$$

$$= S^{*n} f(x,y).$$

For fixed x put $v_x(y) = w(x,y) - u(x)$ so $v_x \in L_1(m)$ and $\int v_x(y) e(y) m(dy) = \int w(x,y) e(y) m(dy) - u(x) \int e(y) m(dy)$ which is zero if we assume $\int e(y) m(dy) = 1$ (this is done as a normalization at the beginning).

Now $h_{nx} \in L_\infty(m)$ with $\| h_{nx} \|_\infty \leqq K \| f \|_\infty$ for almost every x.

$$\left| \int [w(x,y) - u(x)] S^{*n} f(x,y) m(dy) \right| = \left| \int v_x(y) T^{*n} h_{nx}(y) m(dy) \right|$$

$$\leqq \| T^n v_x \|_1 \| h_{nx} \|_\infty \leqq K \| T^n v_x \|_1 \| f \|_\infty \to 0$$

by Theorem 1.1, as $\int v_x e \, dm = 0$.

If we assume that $w(x,y)$ is bounded, then we may use the bounded convergence theorem to obtain

$$\left| \int \int [w(x,y) - u(x)] S^{*n} f(x,y) m(dy) m(dx) \right| \leqq \int \left| \langle v_x, T^{*n} h_{nx} \rangle \right| m(dy) \to 0.$$

Hence by (*) we have proven that if $w(x,y)$ is bounded with $\int \int w e' \, d(m \times m) = 0$ then $S^n w \to 0$ weakly. If w is not bounded, we can find a sequence w_j of bounded functions, with $\int \int w_j e' \, d(m \times m) = 0$, converging to w in L_1-norm, so standard arguments conclude the proof.

COROLLARY 2.1. *Let T be completely mixing having no fixed point. Then for every $w(x,y) \in L_1(m \times m)$ with*

$$\int\int w(x,y)e(x)e(y)m(dx)m(dy) = 0 \textit{ we have } \| S^n w \|_1 \to 0.$$

PROOF. As Theorem 2.1 implies $S^n w \to 0$ weakly in L_1, it is enough to show that S has no fixed point, so Theorem 1.1 applies to S.

Suppose $0 \leqq v(x,y)$ is a fixed point for S. Define $u(x) = \int v(x,y)e(y)m(dy)$. For $g \in L_\infty(m)$ define $f(x,y) = g(x)e(y)$ and by Lemma 1.1 $S^*f(x,y) = T^*g(x)T^*e(y) = T^*g(x)e(y)$. By Fubini's theorem $u(x) \in L_1(m)$, and

$$\langle Tu, g\rangle = \int u(x)T^*g(x)m(dx) = \int\int v(x,y)e(y)T^*g(x)m(dx)m(dy)$$

$$= \int\int v(x,y)S^*f(x,y)d(m \times m) = \langle Sv, f\rangle = \langle v, f\rangle$$

$$= \int\int v(x,y)g(x)e(y)m(dx)m(dy) = \langle u, g\rangle.$$

We obtain thus $Tu = u$ so $u = 0$ a.e. so $0 = \int u(x)e(x)m(dx) = \int\int v e' d(m \times m)$ so $v(x,y) = 0$ a.e. on $Y \times Y$. As an invariant function cannot be supported in the disappearing part, Lemma 1.2 implies that $v(x,y) = 0$ a.e. and S has no fixed points.

We next note that when T has a fixed point $u_0 \in L_1$ with $u_0 > 0$ a.e. it is still true that T is completely mixing if and only if S is completely mixing. This result is known for contractions, by modification of the proof in [2, p. 39].

LEMMA 2.1. *Let T be power bounded and assume $u_0 > 0$ a.e. is a fixed point in L_1. Then for every $u \in L_1$ the averages $(1/N)\sum_{n=1}^N T^n u$ converge in L_1 (to a fixed point). Furthermore, if T is ergodic* (i.e. *there is a unique $e \geqq 0$ in L_∞ with $\int e\, u_0\, dm = 1$* and *$T^*e = e$*), *then* $\lim (1/N) \sum_{n=1}^N T^n u = (\int u\, e\, dm)u_0$ *in L_1.*

We give the well-known arguments of the proof.

As $L_1(u_0 dm)$ is isomorphic to $L_1(m)$ (via the Radon-Nikodym theorem) we may and shall assume $u_0 = 1$. Then T is power bounded in $L_1(m)$ and is also a contraction of L_∞, hence [1, p. 526] a power bounded operator in L_2, to which the mean ergodic theorem can be applied [6, p. 399], so approximations in L_1 yield the desired convergence. If T is ergodic then $v = u - (\int u\, e\, dm)u_0$ is orthogonal to all T^*-invariant functions so $\|(1/N)\Sigma T^n v\|_1 \to 0$.

THEOREM 2.2. *Let T be power bounded having a fixed point $u_0 \in L_1$ with $u_0 > 0$ a.e. Then T is completely mixing if and only if its Cartesian square S is completely mixing.*

PROOF. If S is completely mixing, so is T by the beginning of the proof of Theorem 2.1.

If T is completely mixing, we have (using Lemma 2.1) that $T^n u \to (\int u e \, dm) u_0$ weakly, for every $u \in L_1(m)$, and the arguments in [2, p. 39] can be modified to yield our result.

3. Application to Markov operators

If T is a positive operator on L_1 with $T^*1 = 1$ we call it a *Markov operator.* Theorem 2.1 can be applied to such operators and we can obtain that $T \otimes T \otimes T \cdots \otimes T$ is completely mixing. This holds even if T is conservative and ergodic while $T \otimes T$ is not conservative. Kakutani and Parry [3] have shown that T weak mixing (defined as $T \otimes T$ ergodic) does not imply that $T \otimes T$ is weakly mixing in the absence of a finite invariant measure.

A discussion of the intuitive interpretation of complete mixing is given in [4, §2].

For a conservative Markov operator T we have that if T is completely mixing, so is $T \otimes T \otimes \cdots \otimes T$, because either theorem 2.1 or theorem 2.2 can be applied as T must be ergodic.

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